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THE ACOUSTIC FORCE OF ELECTROSTATIC TYPE

BY

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Abstract. The analysis of the secondary Bjerknes force between two bubbles suggests that this force is analogous to the electrostatic forces. Our paper brings new arguments in support of this analogy. The study we perform is dedicated to resonant acoustic force and in a thermal background to highlight its independence from the angular frequency of inductive waves. Highlighting this analogy will allow us a better understanding of the electrostatic interaction if the electron is modeled as an oscillating bubble in the vacuum.

Keywords: secondary Bjerknes force; acoustic force; acoustic charge.

1. Introduction

The aim of this paper is to bring new arguments for the analogy between the electromagnetic world, *i.e*. the world in which systems and phenomena interact and correlate with electromagnetic waves, and the acoustic world, *i.e*. the world in which systems and phenomena interact and correlate with acoustic waves. The paper highlight that the acoustic forces of electrostatic

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type and the cross section of the acoustic interaction are not dependent of the angular frequency of the waves that induce the oscillations of the bubbles. This property exists in resonant conditions or in interaction with an acoustic background. The analogy between secondary Bjerknes force and the electrostatic force was notified since the early theoretical and experimental study of these forces (Bărbat *et al*., 1999; Bjerknes, 1906; Crum, 1975; Hsiao *et al*., 2001). A general analysis of this analogy was made by Zavtrak (Doinikov, 2003; Zavtrack 1990).

The analysis of the secondary Bjerknes forces between two bubbles reveals that these forces imply the scattering of the inductor acoustic wave, *i.e*. the forcing waves.

The two elastic bubbles absorb energy from the forcing wave and oscillate in volume. These spherical oscillations produce the spherical wave pressure whose intensity decreases as $1/r^2$. The intensity is proportional to the area of the bubble at the bubble surface and r is the distance between the centers of the bubbles. Therefore, the interaction forces between bubbles which is proportional to $1/r^2$ and the product of the scattering cross sections of the two bubbles. The attractive or repulsive forces are depicted as oscillations in the volume of the two spheres which are in phase or phase opposition. The force achieves their maximum at resonance (Bărbat *et al*., 1999; Hsiao *et al*., 2001; Ainslie and Leighton, 2009; Ainslie and Leighton, 2011).

The scattering-absorption phenomenon of the acoustic wave involves a cross section of interaction (Ainslie and Leighton, 2009; Prosperetti, 1977).

The analogy between the electromagnetic world and the acoustic world is also advocated by the experimental and theoretical studies of other phenomena such as: the existence of the acoustic black hole (Simaciu *et al*., 2018), the acoustic Casimir effect (Larraza and Denardo, 1998; Larraza and Denardo, 1999; Bárcenas *et al*., 2004), and the corpuscular wave duality for the acoustic wave packet (Simaciu *et al*., 2015) and for the walking droplet (Couder and Fort, 2006; Harris and Bush, 2014).

This paper is dedicated to the study of the acoustic interaction and also of the electrostatic interaction, mentioned above, in order to highlight their analogy. The study of the two phenomena leads us to a better phenomenological understanding of the microscopic phenomena of the electromagnetic world.

2. The Acoustic Force

2.1. The Secondary Bjerknes Forces

The secondary Bjerknes force, the acoustic force, has been studied in several papers (Bărbat *et al*., 1999; Bjerknes, 1906; Crum, 1975; Hsiao *et al*., 2001; Doinikov, 2003; Prosperetti, 1977). We consider the simpler case study of de T. Bărbat, N. Ashgriz and C-S. Hi Liu.

The expression of acoustic force for two bubbles with different radii is

$$F_B(\omega,r)=\frac{-2\pi R_{01}R_{02}A^2\omega^2\cos\varphi}{r^2\rho\left[\left(\omega^2-\omega_{01}^2\right)^2+4\beta_1^2\omega^2\right]^{1/2}\left[\left(\omega^2-\omega_{02}^2\right)^2+4\beta_2^2\omega^2\right]^{1/2}}, \quad (1)$$

with $A=\varepsilon p_0$, the amplitude of the incident wave pressure.

If the bubbles are identical, $R_{01}=R_{02}=R_0$, then Eq. (1) becomes:

$$F_B(\omega,r)=\frac{2\pi R_0^2}{r^2}\frac{p_0^2\varepsilon^2\omega^2}{\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta^2\omega^2\right]} \quad (2)$$

The acoustic force is dependent on the angular frequency at the limit $\omega_0\to 0$,

$$F_{B0}(r)=\lim_{\omega_0\to 0}F_B(\omega,r)\cong\frac{2\pi R_0^2}{r^2}\frac{p_0^2\varepsilon^2}{\rho\omega^2}, \quad (3)$$

In the limit case $\omega\to\infty$, the acoustic force is zero

$$F_{B\infty}(r)=\lim_{\omega\to\infty}F_B(\omega,r)=0. \quad (4)$$

The acoustic force is independent of angular frequency at resonance of oscillation speeds (Feynman *et al.*, 1964, Ch. 23),

$$F_B(\omega_0,r)=\lim_{\omega\to\omega_0}F_B(\omega,r)\cong\frac{2\pi u^2\left(p_0\varepsilon\right)^2}{r^2\rho\omega_0^4}\cong\frac{2\pi R_0^4}{r^2}\left(\rho u^2\right)\left(\frac{p_0\varepsilon}{p_{eff}}\right)^2. \quad (5)$$

If we write the expression of force (5) as a function of the resonance cross section, $\sigma_{0ac}=\sigma_{0s}=\left(4\pi R_0^2\right)\rho u^2/p_{eff}$ (Eq. (18) from paper (Simaciu *et al.*, arXiv: 1711.03567)), we obtain

$$F_B(\omega_0,r)\cong\frac{\sigma_{0s}^2}{8\pi r^2}\frac{\left(p_0\varepsilon\right)^2}{\rho u^2}, \quad (6)$$

Prior studies made by various authors have highlighted the fact that for identical bubbles, $R_{01}=R_{02}=R_0$, the acoustic force is repulsive only at resonance, $R_{0r}\cong\left(p_{eff}/\rho\omega_0\right)^{1/2}$ (Mettin *et al.*, 1997; Doinikov, 2002; Rezaee *et al.*, 2011; Zhang *et al.*, 2016). Another case where the acoustic force is independent of angular frequency is that one when the bubbles interact with the thermal acoustic background. We study this case in section 3.

2.2. The Electrostatic Force

The electrostatic force (Jackson, 1975, Ch. 1) between two charges, the Coulomb force, is

$$F_C\left(r\right)=\frac{Q_1Q_2}{8\pi\varepsilon_0 r^2}. \tag{7}$$

Since the charge is quantified, $Q=N\left|q_e\right|$, we can express the force according to the electron charge, q_e,

$$F_C\left(r\right)=N_1N_2\frac{q_e^2}{4\pi\varepsilon_0 r^2}=N_1N_2\frac{e^2}{r^2}. \tag{8}$$

In Eq. (8),

$$F_{Ce}\left(r\right)=\frac{q_e^2}{4\pi\varepsilon_0 r^2}=\frac{e^2}{r^2} \tag{9}$$

is the Coulomb force between two electrons in the vacuum. In Eq. (9), e is the electrostatic charge of electron in electrostatic units. This restricts us to focus our analysis to the interaction of two identical bubbles.

3. The Force in a Thermal Acoustic Background

3.1. The Acoustic Forces in a Thermal Acoustic Background

In this section we study the assumption that the acoustic forces of electrostatic type are forces between two bubbles induced by the acoustic thermal background. This background is a compound of acoustic waves with random phase or thermal acoustic radiation at equilibrium (Kittel, 2005, Ch. 5).

The background is assumed to be done by the thermal oscillations of the cavity containing the fluid. In our picture of the background there a lot of identical bubbles and they are at resonance with each other and with the background components.

In order to calculate the force of interaction with the background we will proceed analogous to the way we have adopted when we have expressed the secondary Bjerknes forces between two bubbles. That is, we work out the volume oscillations of a bubble under the action of infinitesimal spherical pressure caused by the acoustic wave background.

The oscillations of the bubble radius is, according to Rayleigh-Plesset relationship (Crum, 1975; Prosperetti, 1977).

$$R_j\ddot{R}_j+\frac{3}{2}\dot{R}_j^2=\frac{1}{\rho}\left(p_{\text{int}}(t)-\frac{2\sigma}{R_j}-\frac{4\mu}{R_j}\dot{R}_j-p_{ext}(t)\right),\ j=1,2, \tag{10}$$

with the external pressure determined by the wave

$$p_{ext}(t)=p_0+A\cos\omega t \tag{11}$$

and p_0 the hydrostatic pressure of the unperturbed fluid.

For small amplitudes, the solution of Eq. (10) is (Bărbat *et al.*, 1999; Prosperetti, 1977)

$$R(t)=R_0\left[1+a\cos\left(\omega t+\varphi\right)\right] \tag{12}$$

with dimensionless amplitude and phase given relationships:

$$a=\frac{A}{\rho R_0^2\left[\left(\omega^2-\omega_0^2\right)^2+4\beta^2\omega^2\right]^{1/2}},\quad \varphi=\arctan\frac{2\beta\omega}{\left(\omega^2-\omega_0^2\right)} \tag{13}$$

and the natural angular frequency given by Eq. (9) and the damping constant given by the Eq. (7) of the paper (Simaciu *et al.*, arXiv: 1711.03567).

The correspondence between the notations adopted by Prosperetti and Barbat is

$$p_0\varepsilon=A,\ x(t)=a\cos\left(\omega t+\varphi\right). \tag{14}$$

In our approach an additional pressure produced by the radial bubble oscillation has to be added to them mentioned above:

$$p'(r,t)=\frac{\ddot{V}}{4\pi r}-\rho\frac{\rho\dot{R}^2}{2}\left(\frac{R}{r}\right)^4\cong\frac{\ddot{V}}{4\pi r}=\frac{R}{r}\left(2\dot{R}^2+R\ddot{R}\right)\cong\frac{R^2\ddot{R}}{r}. \tag{15}$$

Substituting Eq. (12) to Eq. (15) gives

$$p'(r,t)\cong-\frac{\rho\omega^2R_0^3a}{r}\cos\left(\omega t+\varphi\right), \tag{16}$$

with r the distance between the centers of the bubbles.

The expression of secondary Bjerknes force is:

$$\vec{F}_B=\left\langle\vec{F}_{12}\right\rangle=-\left\langle V_2(t)\nabla p_1'(r,t)\right\rangle \tag{17}$$

with

$$V_2(t)=\frac{4\pi R_{02}^3}{3}\left[1+a_2\cos\left(\omega t+\varphi_2\right)\right]^3 \tag{18}$$

and

$$\nabla p_1'(r,t)\cong\frac{\rho\omega^2R_{01}^3a_1}{r^2}\cos\left(\omega t+\varphi_1\right) \tag{19}$$

Substituting Eq. (19) into Eq. (17) then it follows

$$F_B(r)=\langle F_{12}\rangle=-\frac{2\pi\rho\omega^2R_{01}^3R_{02}^3}{r^2}a_1a_2\cos\varphi\Phi\left(a_1,a_2,\varphi=\varphi_2-\varphi_1\right),$$
$$\Phi\left(a_1,a_2,\varphi\right)=1-\frac{a_1a_2}{\cos\varphi}+\frac{1}{4}\left(a_1^2+a_2^2\right)+2a_1a_2\cos\varphi+O\left(a_1^ia_2^j\right),\ i+j\geq 3. \tag{20}$$

Substituting Eq. (13) into Eq. (20), then we are lead to Eq. (1).

In order to complete our approach we will assume that the pressure is an external pressure. This will be done when we will study the interaction of a bubble with the acoustic background. Let remind that waves have a certain angular frequency and a random phase (waves corresponding to an infinitely small angular frequency interval)

$$\delta p_{ext}(t)=p_0+\left(\delta A\right)\cos\left(\omega t+\theta\right). \tag{21}$$

In what it follows we will broach the issue of how to express the acoustic force which can generate an elementary oscillation, *i.e.* an oscillation with elementary amplitude. The thermal background is responsible for this action.

A wave changes the pressure of a liquid according to the relation (Landau and Lifchitz, 1971, Ch. 8, §63)

$$p_w=p\left(\frac{\rho_w}{\rho}\right)^{\gamma}=p\left(1+\frac{\delta\rho_w}{\rho}\right)^{\gamma}=p\left(1+\frac{\dot{q}}{u}\right)^{\gamma}. \tag{22}$$

with $p=p_a=p_0=\rho u^2$ the unperturbed fluid pressure, $\dot{q}$ the speed of the oscillation of the fluid particle and u the wave speed. For some liquid, γ is replace by the adiabatic coefficient for liquids γ_f.

Assuming the notations used for the Bjerknes force, the pressure can be expressed as

$$p_w=p\left(1+\frac{\dot{q}}{u}\right)^{\gamma_f}\cong p_0\left(1+\gamma_f\frac{\dot{q}}{u}\right)=p_0\left[1+\gamma_f\frac{q_0\omega}{u}\cos\left(\omega t-\vec{r}\vec{k}+\theta\right)\right] \tag{23}$$

with the fluid oscillation amplitude q_0. We have adopted in (23) a total phase $\omega t-\vec{r}\vec{k}+\theta$ in order to average over random phase, in the same manner as that one used for the Classical zero point field (Boyer, 1969; Boyer, 1975).

Collating Eq. (23) with the Eq. (11) yields

$$p_{ext}=p_w,\ A=p_0\frac{\gamma_f q_0\omega}{u}=\gamma_f\rho u q_0\omega. \tag{24}$$

For the acoustic background, δA is

$$\delta A=\frac{p_0\gamma_f\omega}{u}h(\omega,\theta)\exp\left[-i\left(\omega t-\vec{k}\vec{r}-\theta\right)\right]d^3k. \tag{25}$$

with θ the random phase and $h(\omega,\theta)$ the wave amplitude.

Therefore, the square of the amplitude determined by a stochastic background is

$$\left\langle A_s^2 \right\rangle = \left(\frac{p_0 \gamma_f}{u} \right)^2 \left\langle \mathrm{Re} \int \omega d^3 k \int \omega' d^3 k' h(\omega) h(\omega') \right. \\ \left. \exp\left[i(\vec{k} - \vec{k}')\vec{r} - i(\omega - \omega')t + i(\theta - \theta') \right] \right\rangle. \tag{26}$$

Calculating the average of the exponential function for the random phase, that is for all values, $\theta \in [0, 2\pi]$, with the same probability, we find out

$$\left\langle \exp\left[i(\vec{k} - \vec{k}')\vec{r} - i(\omega - \omega')t + i(\theta - \theta') \right] \right\rangle = \delta_{\omega\omega'} \delta\left(k - k'\right) \tag{27}$$

According to Dirac function properties, $\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$, then

$$\int \omega d^3 k h(\omega, \theta) \delta(k - k') = \omega' h(\omega', \theta'). \tag{28}$$

Substituting Eqs. (28) and (27) into Eq. (26) the mean square of the amplitude is

$$\left\langle A_s^2 \right\rangle = \left(\frac{p_0 \gamma_f}{u} \right)^2 \int \left(\omega'\right)^2 d^3 k' h^2\left(\omega'\right) \delta_{\omega\omega'} = \left(\frac{p_0 \gamma_f}{u} \right)^2 \int \omega^2 d^3 k h^2\left(\omega\right) = \\ \frac{4\pi \left(p_0 \gamma_f\right)^2}{u^5} \int \omega^4 h^2\left(\omega\right) d\omega, \tag{29}$$

with, $\omega = ku$ and $d^3 k = 4\pi k^2 dk = \left(4\pi\omega^2 d\omega\right) / u^3$.

One can use again the same way of the Classical zero point field approach to express the amplitude function $h(\omega)$, that is to average the stochastic energy density of the background.

The total energy of an oscillator under the action of an elementary wave of a particular angular frequency, is

$$\left\langle E_o \right\rangle = \frac{m}{2} \left\langle \omega^2 q_s^2 \right\rangle = \frac{m}{2} \left\langle \mathrm{Re} \int \omega d^3 k \int \omega' d^3 k' h(\omega) h(\omega') \exp\left[i(\vec{k} - \vec{k}')\vec{r} - \right.\right. \\ \left.\left. i(\omega - \omega')t + i(\theta - \theta') \right] \right\rangle = \frac{m}{2} \int \omega^2 h^2(\omega) d^3 k = \frac{2\pi m}{u^3} \int \omega^4 h^2(\omega) d\omega. \tag{30}$$

Energy density of a stochastic wave is

$$w_s = n\left\langle E_o \right\rangle = \frac{2\pi\left(nm\right)}{u^3} \int \omega^4 h^2(\omega) d\omega = \frac{2\pi\rho}{u^3} \int \omega^4 h^2(\omega) d\omega \tag{31}$$

with n the fluid volume density of the particles and $\rho = nm$ the fluid density.

From the (27), the spectral density of the electromagnetic background is

$$\rho(\omega) = \frac{dw_s}{d\omega} = \frac{2\pi\rho}{u^3}\omega^4 h^2(\omega). \tag{32}$$

This spectral density has a temperature dependence of the form

$$\rho(\omega,T) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\exp\left[\hbar\omega/(kT)\right]-1} \tag{33}$$

and for the zero temperature of the electromagnetic background (Boyer, 1969)

$$\rho(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3}. \tag{34}$$

For the acoustic background, these spectral densities are:

$$\rho(\omega,T) = \frac{\hbar\omega^3}{2\pi^2 u^3}\frac{1}{\exp\left[\hbar\omega/(kT)\right]-1}, \tag{35}$$

$$\rho(\omega) = \frac{\hbar\omega^3}{4\pi^2 u^3}, \tag{36}$$

because the acoustic waves travel with speed u and are they are not polarized. In fluid, waves are longitudinal or compressive, therefore an additional factor $1/2$ is present in Eq. (36).

Comparing the relations (32) and (35), then it results

$$h(\omega,T) = \frac{\hbar^{1/2}}{\left[4\pi^3\rho\omega\left(\exp\frac{\hbar\omega}{kT}-1\right)\right]^{1/2}}. \tag{37}$$

We will use (37) in order to estimate the force determined by the acoustic stochastic background.

If the pressure of the stochastic wave is given by Eq. (25), then the amplitude of the bubble oscillation ($R(t) = R_0\left[1+\delta a(t)\right]$) vary as

$$\delta a(t) = \frac{\delta A(t)}{\rho R_0^2\left[\left(\omega^2-\omega_0^2\right)^2+4\beta^2\omega^2\right]^{1/2}} = \frac{\gamma_f p_0\omega h(\omega,\theta)\exp\left[-i\left(\omega t-\vec{k}\vec{r}-\theta-\varphi\right)\right]d^3k}{uR_0^2\left[\left(\omega^2-\omega_0^2\right)^2+4\beta^2\omega^2\right]^{1/2}} \tag{38}$$

with this relation one can express the volume

$$\delta V_2(t)=\frac{4\pi R_{02}^3}{3}\left[1+\delta a_2(t)\right]^3 \cong 4\pi R_{02}^3\delta a_2(t)=$$
$$\frac{4\pi\gamma_f R_{02}p_0\omega h(\omega,\theta)\exp\left[-i\left(\omega t-\vec{k}\vec{r}-\theta-\varphi_2\right)\right]d^3k}{u\left[\left(\omega^2-\omega_{02}^2\right)^2+4\beta_2^2\omega^2\right]^{1/2}}. \tag{39}$$

and the pressure

$$\delta p_1'(r,t)\cong-\frac{\rho\omega'^2R_{01}^3\delta a_1(t)}{r}=$$
$$\frac{\rho\omega'^2R_{01}}{r}\frac{u\omega' h(\omega',\theta')\exp\left[-i\left(\omega' t-\vec{k}'\vec{r}-\theta-\varphi_1\right)\right]d^3k'}{\left[\left(\omega'^2-\omega_{01}^2\right)^2+4\beta_1^2\omega'^2\right]^{1/2}}. \tag{40}$$

Then Eq. (17), with $\delta V_2(t)$ and $\delta p_1'(r,t)$, expressed above, becomes

$$\delta\vec{F}_B=-\delta V_2(t)\nabla\left[\delta p_1'(r,t)\right]. \tag{41}$$

We will assume that the origin of the reference system is in the centre of the bubble 1 and the second bubble has the position vector $\vec{r}$. Substituting Eq. (39) and Eq. (40) in Eq. (41), it leads to

$$\delta F_B(r)=\frac{-4\pi R_{01}R_{02}\left(p_0\gamma_f\right)^2}{r^2\rho u^2}\frac{\left\{\omega h(\omega,\theta)\exp\left[-i\left(\omega t-\vec{k}\vec{r}-\theta-\varphi_2\right)\right]d^3k\right\}}{\left[\left(\omega^2-\omega_{02}^2\right)^2+4\beta_2{}^2\omega^2\right]^{1/2}}\times$$
$$\frac{\left\{\omega'^3h(\omega',\theta')\exp\left[-i\left(\omega' t-\vec{k}'\vec{r}-\theta-\varphi_1\right)\right]d^3k'\right\}}{\left[\left(\omega'^2-\omega_{01}^2\right)^2+4\beta_1'^2\omega'^2\right]^{1/2}}. \tag{42}$$

For the case when the bubbles are identical, Eq. (42) becomes

$$\delta F_B(r)=\frac{-4\pi R_0^2\left(p_0\gamma_f\right)^2}{r^2\rho u^2}\frac{\left\{\omega h(\omega,\theta)\exp\left[-i\left(\omega t-\vec{k}\vec{r}-\theta-\varphi\right)\right]d^3k\right\}}{\left[\left(\omega^2-\omega_0^2\right)^2+4\beta^2\omega^2\right]^{1/2}}\times$$
$$\frac{\left\{\omega'^3h(\omega',\theta')\exp\left[-i\left(\omega' t-\vec{k}'\vec{r}-\theta-\varphi\right)\right]d^3k'\right\}}{\left[\left(\omega'^2-\omega_0^2\right)^2+4\beta'^2\omega'^2\right]^{1/2}}. \tag{43}$$

By integrating and performing the average for random phase in Eq. (43), the secondary Bjerknes force has the form

$$F_B(r) = \frac{-4\pi R_0^2 (p_0\gamma_f)^2}{r^2 \rho u^2} \mathrm{Re}\left\langle \iint \frac{\left\{\omega h(\omega,\theta)\exp\left[-i\left(\omega t - \vec{k}\vec{r} - \theta - \varphi\right)\right] d^3k\right\}}{\left[\left(\omega^2 - \omega_0^2\right)^2 + 4\beta^2\omega^2\right]^{1/2}} \times \right.$$

$$\left. \frac{\left\{\omega'^3 h(\omega',\theta')\exp\left[i\left(\omega' t - \vec{k}'\vec{r} - \theta - \varphi\right)\right] d^3k'\right\}}{\left[\left(\omega'^2 - \omega_0^2\right)^2 + 4\beta'^2\omega'^2\right]^{1/2}} \right\rangle = \frac{-4\pi R_0^2 (p_0\gamma_f)^2}{r^2 \rho u^2} \times \quad (44)$$

$$\int_{k_m}^{k_M} \frac{\omega^4 h^2(\omega,\theta) d^3k}{\left[\left(\omega^2 - \omega_0^2\right)^2 + 4\beta^2\omega^2\right]} = \frac{-(4\pi)^2 R_0^2 (p_0\gamma_f)^2}{r^2 \rho u^5} \int_{\omega_m}^{\omega_M} \frac{h^2(\omega,\theta)\omega^6 d\omega}{\left[\left(\omega^2 - \omega_0^2\right)^2 + 4\beta^2\omega^2\right]}.$$

The integration limits $\omega_M = 2\pi u/\lambda_m = \pi u/a \cong \omega_D$ and $\omega_m = 2\pi u/\lambda_M = \pi u/L$ correspond to the minimum wavelength, $\lambda_m = 2a$ and the maximum wavelength $\lambda_M = 2L$. The maximum angular frequency is approximate the Debye angular frequency, $\omega_D = (u/a)(6\pi^2)^{1/3}$ (Kittel, 2005, Ch. 5).

In our above relationships, a is the average distance between the fluid particles and L is the linear dimension of the container that delimits the fluid.

Substituting Eq. (37) into Eq. (44), we finally find

$$F_B(r) = \frac{-4(p_0\gamma_f)^2 R_0^2 \hbar}{\pi\rho^2 u^5 r^2} \int_{\omega_m}^{\omega_M} \frac{\omega^5 d\omega}{\left(\exp\frac{\hbar\omega}{kT} - 1\right)\left[\left(\omega^2 - \omega_0^2\right)^2 + 4\beta^2\omega^2\right]}. \quad (45)$$

3.2. The Acoustic Force of the Electrostatic Type and the Acoustic Charge

In order to estimate analytically the expression (45) of the force, we calculate the integral using the saddle-point method (Puthoff, 1987; Feynman *et al.*, 1964, Ch. 23). In Eq. (45), the integral has a maximum for $\omega \cong \omega_0$. If we make the change of variables $\omega - \omega_0 = y$, the integral becomes

$$I_B = \frac{\omega_0^5}{4\omega_0^2\left(\exp\frac{\hbar\omega_0}{kT} - 1\right)} \int_{y_m}^{y_M} \frac{-dy}{y^2 + \beta_0^2} =$$
$$\frac{\omega_0^3}{4\beta_0\left(\exp\frac{\hbar\omega_0}{kT} - 1\right)}\left[\arctan\frac{\omega_M - \omega_0}{\beta_0} - \arctan\frac{\omega_m - \omega_0}{\beta_0}\right], \quad (46)$$

with

$$\beta_0 = \beta(\omega_0) = 2\frac{\mu + \mu_{th}}{\rho R_0^2} + \frac{\omega_0^2 R_0}{2u} \cong \frac{\omega_0^2 R_0}{2u}. \tag{47}$$

If $\omega_M \gg \omega_0, \omega_m \ll \omega_0$, we can approximate: $\omega_M - \omega_0 \cong \omega_M = 2\pi u/\lambda_m = \pi u/a \to \infty$, and $\omega_m - \omega_0 \cong -\omega_0$. With these approximations, Eq. (46) becomes

$$I_B \cong \frac{\omega_0^3}{4\beta_0\left(\exp\frac{\hbar\omega_0}{kT} - 1\right)}\left(\frac{\pi}{2} + \arctan\frac{\omega_0}{\beta_0}\right) \cong \frac{\pi u\omega_0}{4R_0\left(\exp\frac{\hbar\omega_0}{kT} - 1\right)}\left(1 + \frac{2}{\pi}\arctan\frac{2u}{R_0\omega_0}\right) \cong \frac{\pi u\omega_0}{4R_0\left(\exp\frac{\hbar\omega_0}{kT} - 1\right)}. \tag{48}$$

Substituting Eq. (48) into Eq. (45), it follows

$$F_B(R_0, \omega_0, T, r) \cong \frac{R_0\hbar\omega_0}{\left(\exp\frac{\hbar\omega_0}{kT} - 1\right) r^2}\left(\frac{p_0\gamma_f}{\rho u^2}\right)^2. \tag{49}$$

The expression of the acoustic force magnitude given by Eq. (49) can be put under the form $F_a = e_a^2/r^2$ in order to suggest the existence of an acoustic charge (equivalent of charge in electrostatic units), e_a. The square of the acoustic charge is

$$e_a^2 = \frac{R_0\hbar\omega_0}{\left(\exp\frac{\hbar\omega_0}{kT} - 1\right)}\left(\frac{p_0\gamma_f}{\rho u^2}\right)^2 = \frac{\hbar u}{\exp\frac{\hbar u}{kTR_0}\left(\frac{p_{eff}}{\rho u^2}\right)^{1/2} - 1}\left(\frac{p_0\gamma_f}{\rho u^2}\right)^2\left(\frac{p_{eff}}{\rho u^2}\right)^{1/2}. \tag{50}$$

Depending on the ratio between $\hbar\omega_0$ and kT, we get the following expressions of the square of the acoustic charge:

$$e_a^2 = kTR_0\left(\frac{p_0\gamma_f}{\rho u^2}\right)^2, \hbar\omega_0 << kT; \tag{51}$$

$$e_a^2 = \frac{R_0\hbar\omega_0}{(e-1)}\left(\frac{p_0\gamma_f}{\rho u^2}\right)^2 = \frac{\hbar u}{(e-1)}\left(\frac{p_0\gamma_f}{\rho u^2}\right)^2\left(\frac{p_{eff}}{\rho u^2}\right)^{1/2}, \hbar\omega_0 \cong kT; \tag{52}$$

$$e_a^2 = \frac{R_0\hbar\omega_0}{\exp\frac{\hbar\omega_0}{kT}}\left(\frac{p_0\gamma_f}{\rho u^2}\right)^2 = \frac{\hbar u}{\exp\frac{\hbar u}{kTR_0}\left(\frac{p_{eff}}{\rho u^2}\right)^{1/2}}\left(\frac{p_0\gamma_f}{\rho u^2}\right)^2\left(\frac{p_{eff}}{\rho u^2}\right)^{1/2}, \hbar\omega_0 >> kT. \tag{53}$$

According to these equations, the acoustic charge depends on the bubble and liquid parameters.

The relationship (50) highlights the existence of a maximum acoustic charge

$$e_{am}^2 = \hbar u. \quad (54)$$

This happens similarly to the electromagnetic world, where there is a maximum charge of interaction $e_m^2 = \hbar c$, such that $\hbar c / e^2 = e_m^2 / e^2 \cong 137$ (Jackson, 1975, Subch. 6.12).

4. Conclusions

Previous papers on the issue of the analogy between the electrostatic and the acoustic forces were based on the fact that both forces depend inverse proportional to the square of the distance between the physical systems in interaction. Also the analogy have taken account of the symmetrically dependence of the forces on the parameters of the two systems, the particles and the bubbles, (e_1, e_2) and (R_{01}, R_{02}), and the inductive acoustic wave parameters $(\varepsilon p_0, \omega)$. For this reason, the acoustic force was considered to be similar to the gravitational force between two masses (Bărbat *et al*., 1999).

We showed in this paper that the acoustic force caused by the scattering process $(\beta_{ac} >> \beta_\mu + \beta_{th})$ cannot be analogous to the gravitational force because it is repulsive, $\varphi \in (\pi/2, \pi]$, and also attractive, $\varphi \in [0, \pi/2)$. This situation is similar to the electrostatic interaction.

The acoustic force given by the Eq. (52) is not quite an electrostatic type force because in the fluid the bubbles have different radii and no quantification of the radii was observed. It is necessary to study the formation and evolution of vapour bubbles (without gas) to find out the conditions in which they have various values of very close radii. In electrostatics, systems can have various electrical charges through the accumulation of elementary charges, which are the charges carried by the electron and the proton. In our paper, we studied the limited case of the interaction of two identical bubbles. For identical bubbles, we identified the existence of an acoustic charge and an acoustic cross section (see the paper (Simaciu *et al*., arXiv: 1711.03567)). The acoustic charge also depends on the amplitude of the forcing wave. The acoustic charge, are not related of the angular frequency for angular frequency close to the natural angular frequency that is at resonance. The acoustic charge is dependent on the magnitude of the forcing wave. In order to eliminate the dependencies we had two options to consider: the two bubbles interact with the background of the thermal radiation or the two bubbles interact with the background created by other identical oscillating bubbles in the container. Adopting the first option, we have obtained an acoustic charge and a scattering cross section analogous to electrostatic ones. We can say that in this former case the acoustic interaction is analogous to the electrostatic interaction. Applying this approach of the electrostatic interaction to the electromagnetic world, *i.e.* for the electron, one can be obtained a good evidence for the connection between the blackbody radiation, the relativity, and the discrete charge in classical electrodynamics (Boyer, 2007). The latter case will be analysed in a further paper.

We appreciate that the approach would be complete when a theoretical and an experimental evidence of a magnetic type interaction for oscillating bubbles in the translational motion would be revealed. Also, when should be revealed a magnetic field type around a bubble which performs a rotation with constant angular velocity.

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FORȚA ACUSTICĂ DE TIP ELECTROSTATIC

(Rezumat)

Analiza forței secundare Bjerknes între două bule sugerează că această forță este analogă forțelor electrostatice. Lucrarea noastră aduce noi argumente în sprijinul acestei analogii. Studiul pe care îl efectuăm este dedicat forței acustice la rezonanță și într-un fond acustic termic pentru a evidenția independența sa de frecvența unghiulară a undelor inductive. Evidențierea acestei analogii ne va permite o mai bună înțelegere a interacțiunii electrostatice dacă electronul este modelat ca o bulă oscilantă în vacuum.